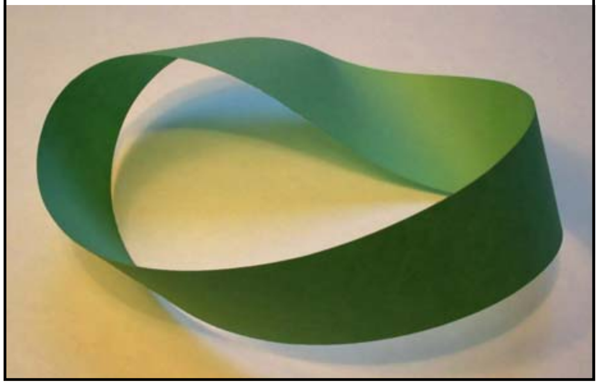
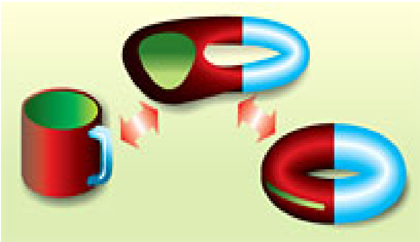
# Topology and Physics



# Topology

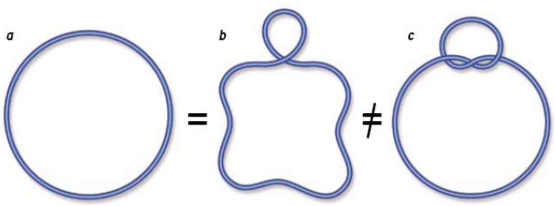
- Topology is a branch of mathematics concerned with geometric configurations that are unchanged by *elastic deformations* or *twists*.
- A topologist cannot tell the difference between a coffee cup and a donut!


- One can be *continuously deformed* into the other by stretching and indenting the surface without tearing it
- Topological equivalence can only be destroyed by a drastic change such as tearing or gluing parts together.



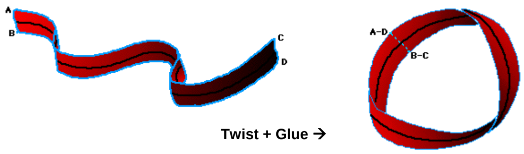
# Coffee Cup = Donut

# Topology and Knot Theory




Has applications to quantum computing and DNA replication

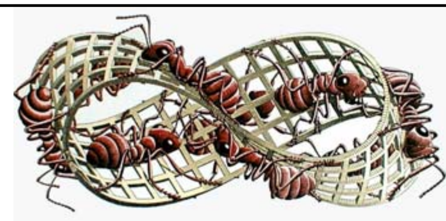
# Möbius Strip




- **the Möbius Strip has only one side and one edge!**

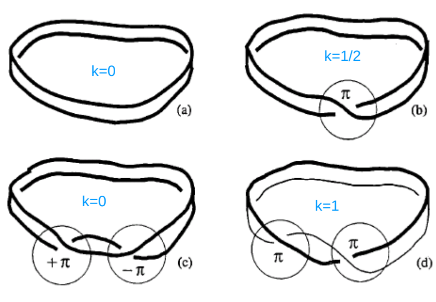
## One sided? One edge?

- (1) start midway between the "edges" of a Möbius Strip and draw a line down its center; continue the line until you return to your starting point without lifting the pen.
  Did you ever cross an edge?
- (2) Next, hold the edge of a Möbius Strip against the tip of a felt-tipped highlighter pen. Color the edge of the Möbius Strip by holding the highlighter still and just rotating the Mobius Strip around.
  Were you able to colour the entire edge?
- (3) Now, with scissors cut the Möbius Strip along the center line that you drew. Then draw a center line around the resulting band, and cut along it.
  Did you predict what would happen?



- The famous artist, M.C. Escher, used mathematical themes in some of his work, including a Möbius parade of ants
- Giant Möbius Strips have been used as conveyor belts (to make them last longer, since "each side" gets the same amount of wear) and as continuous-loop recording tapes (to double the playing time).

## Twist = $2\pi k$ **($2\pi = 360^0$)**




k= 0,±1/2,±1,… is the "charge" and labels topologically distinct states

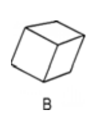
## Quantum Numbers

- The different values of "k" can be used to classify the topology of the strips
- These are conserved quantities like "*charges*"
- half integer strips are *one sided* and integer strips are *two sided*
- In quantum mechanics, angular momentum is conserved and can take *integer* or *half-integer* values
- Half-integer values describe fermions (electrons, protons, neutrons, quarks)
- Integral values describe bosons ($^4$He, photons)
- In quantum mechanics we can classify particles in a similar way to that we used to classify strips

## Symmetry

•What distinguishes different phases of matter?
•Phases differ in their symmetry
•Which is more symmetric?

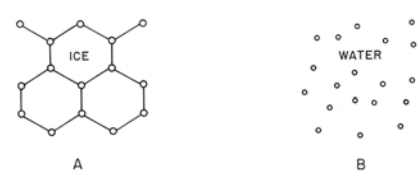

**A sphere**




**Or a cube?**

**A cube breaks rotational symmetry**
**=> some directions are more equal than others**

## Which is more symmetric?

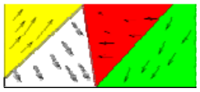



**•Ice can be rotated by $120^0$ or $240^0$**
**•Breaks translational symmetry**
**=> can only be shifted by special distances**

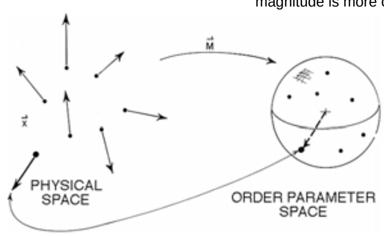
**•No symmetry at all!**
**•Complete rotational and translational symmetry**

**•How do we tell if two materials differ by a symmetry?**
**•Can we change one into the other smoothly by changing temperature?**
**•Ice turns to water at a phase transition**

## What is the order parameter for a magnet?



- At each point (x,y,z) we have a magnetization *vector* **M**(**x**)
- In becoming a magnet, the material has broken rotational symmetry
- Direction varies from point to point but magnitude is more or less fixed



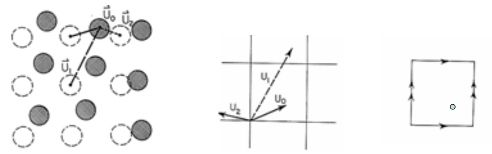

Different points related by rotations

Vector with 3 components

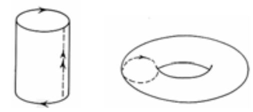
Map the directions of **M** onto a sphere

## Two dimensional crystal

- Important degrees of freedom associated with broken translational order
- Consider a deformation described by displacement vectors **U**(**x**)
- However which "ideal" atom is associated with which "real" atom?



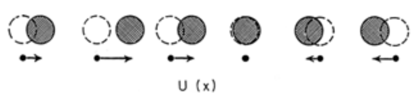
- **U**(**x**) = **U**(**x**) + ma **i** + na **j** *all are equivalent points in a square*
- Hence the order parameter U(x) is a square with periodic boundary conditions



Order parameter space is a torus

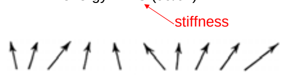
## Excitations

- Crystals are rigid because of the broken translational symmetry
- Uniform displacements costs no energy
- Energy depends on derivatives or gradients of displacements
- Long wavelength waves have low frequencies =>sound waves
- Broken symmetry=> low frequency excitations (Goldstone theorem)




Long wavelength modulations of the order parameter cost an energy ~ $\frac{1}{2}C\,(du/dx)^2$

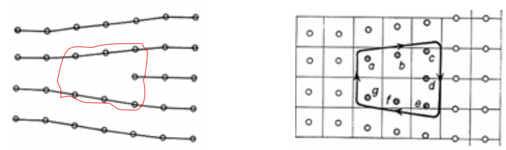
stiffness



Magnetic waves or spin waves

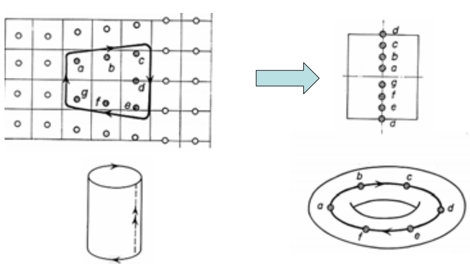
## Topological Defects

- A defect is a tear in the order parameter field
- Consider the two dimensional crystal with an extra row of atoms called a dislocation

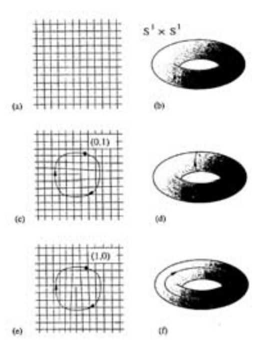


- Away from the middle there are small distortions
- Can we repair the defect by simple rearrangements?
- No! the effect of the defect extends a long way
- Consider a closed path surrounding the defect and count the rows crossed
- For any path enclosing the core there will be an extra row on the right

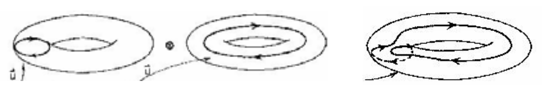
## Order Parameter Space



- Moving around the loop corresponds to a loop in the order parameter space
- Deforming the atoms slightly does not change the number of times the loop winds around the hole => winding number describes the defect
- Loop cannot be shrunk to a point => topological space is not simply connected

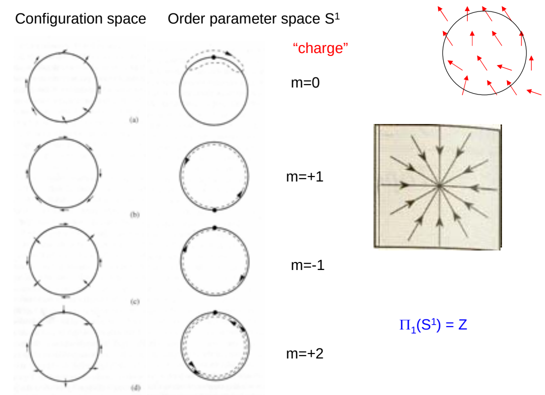


# Homotopy Classes



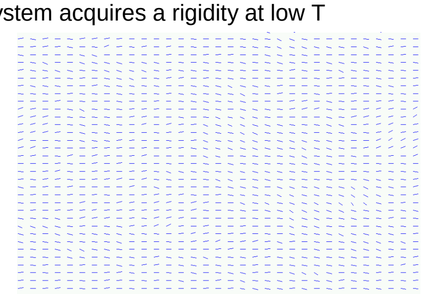
- If the order parameter field does not wind around the torus, it can be smoothly deformed back to a uniform state
- If it winds around the hole in the torus or through it, then it cannot be continuously deformed back to the uniform state
- Two integers needed to describe the defect → (m,n) *Burger's vector*
- First homotopy *group* $\Pi_1(T^2) = Z \times Z$
- Two loops are equivalent if they can be twisted into one another
  → form equivalence classes

2 component spin model <u>or</u> xy model <u>or</u> planar model

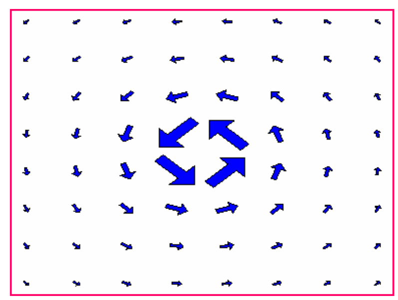



# d=2 xy model

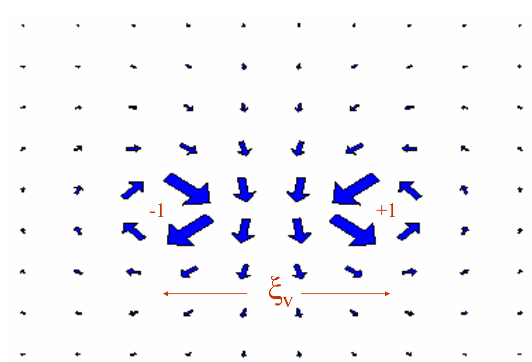
- Kosterlitz and Thouless (1973) predicted that the transition is a *defect unbinding transition*
- system acquires a rigidity at low T



Charge=0



**Single vortex (charge= +1) costs infinite energy to create**




**Pair of vortices (total charge zero) costs a finite amount of energy**